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FACTORIZATION OF THE DISCRETE NOISE COVARIANCE MATRIX FOR PLANS (U)

by

J. Chris McMillan

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FACTORIZATION OF THE DISCRETE NOISE COVARIANCE MATRIX FOR PLANS(U)

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J. Chris McMillan

*Navigation and Integrated System Section
Electronics Division*

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ABSTRACT

The PLANS (Primary Land Arctic Navigation System), developed at DREO, optimally integrates a directional gyro/gyrocompass, an odometer, a 3-axis strapdown magnetometer, a GPS receiver, a Transit receiver, a baroaltimeter and a digital terrain elevation map, for the purpose of navigating a land vehicle in the Canadian Arctic under potentially adverse conditions. This report derives the exact form of the discrete driving noise covariance matrix Q_k which is needed to propagate the covariance matrix in the Kalman filter used by PLANS. It is shown that the exact Q_k does not have a Cholesky UDU^T decomposition. However, a good approximation is shown to have the necessary decomposition for use in the Biermann-Agee-Turner formulation of the Kalman filter. This approximate decomposition is then found. A general result on the preservation of block diagonal form under UDU^T decomposition is also proven.

RÉSUMÉ

Le système de navigation terrestre PLANS, conçu au CRDO, intègre de façon optimale un gyroscope à deux modes (gyrocompas et directionnel), un odomètre, une sonde magnétique à trois axes, un récepteur GPS, un récepteur TRANSIT, un altimètre barométrique ainsi qu'une carte d'élévation numérique. PLANS a été conçu pour opérer dans l'arctique canadien à bord de véhicules terrestres. Ce rapport présente la formulation exacte de la matrice de covariance Q_k nécessaire pour la propagation de la matrice de covariance du filtre Kalman utilisé par PLANS. Il est démontré que Q_k ne peut être décomposé selon la méthode Cholesky UDU^T . Il est toutefois démontré qu'on peut obtenir d'une bonne approximation la décomposition nécessaire pour utiliser la formulation Biermann-Agee-Turner du filtre Kalman. Cette décomposition approximative est démontrée. Il est aussi démontré que la décomposition UDU^T préserve la forme diagonale.



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EXECUTIVE SUMMARY

PLANS (Primary Land Arctic Navigation System) is a multi-sensor integrated navigation system developed at DREO. PLANS employs an 8 state Kalman filter to optimally integrate the sensor data from a Transit receiver, a GPS receiver, a gyrocompass/directional gyro, an odometer, a magnetometer and a baroaltimeter. In the process of deriving and implementing the Kalman filter equations, one of the many matrices that must be found is the discrete process noise covariance matrix Q_k (also known as the driving noise covariance).

Initially, as is common practice, an approximation was used to evaluate this Q_k . During a detailed analysis of simulation results, the behaviour of the PLANS position error covariance matrix, P , under propagation (i.e. without position measurements from Transit or GPS) came under suspicion. This behaviour is governed solely by the state transition matrix $\Phi(t, \Delta t)$ (at time t over an interval Δt) and the driving noise covariance, Q_k . The discrete driving noise covariance matrix, Q_k , over the interval Δt , is itself defined by the continuous driving noise power spectral density matrix, Q , and the state transition matrix, $\Phi(t, \Delta t)$. Therefore these matrices came under special scrutiny. Since $\Phi(t, \Delta t)$ was already exact this then led to the desire for a more exact Q_k . The purpose of this report is therefore to derive the "exact" form of Q_k (and find its Cholesky decomposition for use in PLANS).

Since PLANS employs a "square root" formulation of the Kalman filter equations for improved numerical stability, it is therefore necessary to find the Cholesky UDU^T decomposition of Q_k (where U is an upper triangular matrix and D is diagonal). For the original approximation, Q_k was diagonal so that its decomposition was trivial. With the more exact Q_k this is no longer the case. Furthermore the exact Q_k is not constant, so that the use of a numerical routine to find its decomposition would require considerable computation, making an explicit decomposition highly desirable.

It is proven in this report that in general UDU^T decomposition preserves block diagonal form, and therefore that the process of finding an exact decomposition of a large block diagonal matrix can be reduced to the much simpler problem of decomposing the smaller blocks. This is then applied to the Q_k for PLANS, so that instead of having to decompose an 8x8 matrix, it is only necessary to decompose three 1x1 matrices (which is trivial), one 2x2 matrix and one 3x3 matrix. The 2x2 matrix is easily decomposed exactly. The 3x3 matrix however causes some difficulty. Its UDU^T decomposition is easily enough found, however it is not a Cholesky decomposition because its D component has negative diagonal elements. This is not particularly surprising since the 3x3 matrix is not positive definite and therefore the existence of its Cholesky decomposition is not guaranteed.

It is thus shown that a Cholesky decomposition of the exact Q_k is not possible, and an approximation is still required. An approximation is found which is exact for all but a few of the small off-diagonal terms of Q_k .

As it turned out, the "suspicious" behaviour of the error state covariance matrix P was not due

to the inexactness of Q_k , but in fact could be explained by closer examination of the effect of Φ . Although in hindsight this behaviour seems obviously correct, the suspicion arose because of the intuitive expectation that the position uncertainty (represented by bottom right 2x2 block of P , since the position error states are the last two elements of the state vector) should increase, or at least not decrease, in the absence of position measurements. Although it is true that the covariances of most elements of the state vector behave in this way (since they are independent Markov processes), it is not generally true for the position covariance. This is because of a geometric effect which can lead to a cancellation of errors in some situations, such that the position covariance locally decreases. In the case of PLANS this could be due to the effect of the speed and heading errors while returning to the starting point partially cancelling the errors accumulated during the outbound portion of the trip (a perfectly constant heading and speed error would perfectly cancel if movement were on a plane). Since this geometric effect is correctly modelled in the PLANS Kalman filter, the covariance behaves accordingly.

Although the "problem" that this effort was intended to solve turned out not to be a problem, a more exact form of the driving noise covariance is of course desirable in any case, and the general result on preservation of block diagonal form under decomposition is also quite useful.

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1. INTRODUCTION

Reference [1] describes the multi-sensor integrated navigation system called PLANS (Primary Land Arctic Navigation System), along with the 8 state Kalman filter used for integration of the sensor data. In the process of deriving and implementing the Kalman filter equations, one of the many matrices that must be found is the discrete process noise covariance matrix, Q_k (also known as the driving noise covariance). In reference [2] an approximation was used to evaluate this Q_k . During a detailed analysis of simulation results, the behaviour of the PLANS position error covariance matrix, P , under propagation (i.e. without position measurements from Transit or GPS) came under suspicion. This behaviour is governed solely by the state transition matrix, $\Phi(t\Delta t)$, and the driving noise covariance, Q_k , as follows:

$$P_{t+\Delta t} = \Phi(t\Delta t) P_t \Phi^T(t\Delta t) + Q_k \quad (1)$$

where the discrete driving noise covariance matrix, Q_k , over the interval Δt is defined by the continuous driving noise power spectral density matrix, Q , and the state transition matrix, $\Phi(t\Delta t)$, as follows (see for example reference [5]):

$$Q_k = \int_0^{\Delta t} \Phi(t, \tau) Q \Phi^T(t, \tau) d\tau \quad (2)$$

Therefore these matrices came under special scrutiny, which then led to the desire for a more exact Q_k , since $\Phi(t, \tau)$ was already exact. The purpose of this report is therefore to derive the "exact" form of Q_k and find its Cholesky decomposition for use in PLANS.

Since PLANS employs a "square root" formulation of the Kalman filter equations for improved numerical stability (see reference [3]), it is therefore necessary to find the Cholesky UDU^T decomposition of Q_k (where U is an upper triangular matrix and D is diagonal). For the original approximation, Q_k was diagonal so that its decomposition was trivial. With the more exact Q_k this is no longer the case. Furthermore the exact Q_k is not constant, so that use of a numerical routine to find its decomposition would require considerable computation, making an explicit decomposition highly desirable.

In fact it turns out that a Cholesky decomposition of the exact Q_k is not possible, as will be shown below, and an approximation is still required. This approximation however only involves some of the small off-diagonal terms, and is exact for most terms of Q_k .

As it turned out, the "suspicious" behaviour of the error state covariance matrix P was not due to the inexactness of Q_k , but in fact could be explained by closer examination of the effect of Φ . Although in hindsight this behaviour seems obviously correct, the suspicion arose because of the intuitive expectation that the position uncertainty (represented by bottom right 2×2 block of P , since the position error states are the last two elements of the state vector) should increase, or at least not decrease, in the absence of position measurements. Although it is true that the covariances of most elements of the state vector behave in this way (since they are independent Markov processes), it is not generally true for the position covariance. This is because of a geometric effect which can lead to a cancellation of errors in some situations, such that the position covariance locally decreases. In the case of PLANS this could be due to the effect of the speed and heading errors while returning to the starting point partially cancelling the errors accumulated during the outbound portion of the trip (a perfectly constant heading and speed error would perfectly cancel if movement were on a plane). Since this geometric effect is correctly modelled in the PLANS Kalman filter, the covariance behaves accordingly.

Although the "problem" that this effort was intended to solve turned out not to be a problem, a more exact form of the driving noise covariance is of course desirable in any case, and the general result of Appendix A is also quite useful.

2. THE CONTINUOUS Q MATRIX FOR PLANS

Since the discrete Q_k matrix is found by integrating the continuous power spectral density matrix, Q , folded with the propagation matrix $\Phi(t, \tau)$, as in (2), we must first examine these two matrices.

As was shown in reference [2] the PLANS error state vector contains 5 independent first order Markov processes and three states which are derived from the integrals of these Markov processes. Thus the continuous power spectral density matrix Q has diagonal elements for each of these Markov processes, as follows:

$$Q = \begin{pmatrix} q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3)$$

where the q_i are the constant values of the PSD's (power spectral densities) of the white driving noise for each of the individual Markov states. These can be expressed in terms of the standard Markov process error model parameters (correlation time T_i and steady state covariance p_i) as follows (see any standard text, such as [5]):

$$q_i = 2p_i/T_i \quad (4)$$

The PLANS error state transition matrix, $\Phi(t, \tau)$, is also derived in reference [2], where it is shown to be (at time t , over the interval τ):

$$\Phi(t, \tau) = \begin{pmatrix} e^{-\tau/T1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\tau/T2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-\tau/T3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\tau/T4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T4(1-e^{-\tau/T4}) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\tau/T6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tau S \sin \theta & \tau S \cos \theta & 1 & 0 \\ 0 & 0 & 0 & 0 & \tau S \cos \theta & \tau S \sin \theta & 0 & 1 \end{pmatrix} \quad (5)$$

where the T_i are Markov process correlation times (constants), $S(t)$ is the vehicle speed, $\theta(t)$ is the vehicle heading and τ is the propagation period.

3. THE DISCRETE Q_k MATRIX FOR PLANS

The discrete driving noise covariance matrix, Q_k , over the interval Δt , is defined by the continuous driving noise power spectral density matrix, Q , and the state transition matrix, $\Phi(t, \tau)$, as shown in equation (2):

$$Q_k = \int_0^{\Delta t} \Phi(t, \tau) Q \Phi^T(t, \tau) d\tau \quad (6)$$

where from equations (3) and (5) we can see that the integrand is:

$$\Phi Q \Phi^T =$$

$$\begin{pmatrix} e^{-\tau/T1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\tau/T2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-\tau/T3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\tau/T4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T4(1-e^{-\tau/T4}) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\tau/T6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tau S \sin \theta & \tau S \cos \theta & 1 & 0 \\ 0 & 0 & 0 & 0 & \tau S \cos \theta & \tau S \sin \theta & 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 & & & & & & & \\ & q_2 & & & & & & \\ & & q_3 & & & & & \\ & & & q_4 & & & & \\ & & & & 0 & & & \\ & & & & & q_6 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix} \Phi^T \quad (7)$$

Now from this we can already see that Q_k will have the same block diagonal form as Φ , namely three 1-blocks and a 5-block. As shown in Appendix A, this block diagonal form is also preserved under UDU^T decomposition. Therefore the top three 1-blocks have trivial decompositions, since they are already diagonal. Thus for $i=1,2,3$ we have:

$$Q_k(i,i) = \int_0^{\Delta t} e^{-\tau/T_i} q_i e^{-\tau/T_i} d\tau \quad (8)$$

$$= \left. \frac{-T_i}{2} q_i e^{-2\tau/T_i} \right|_0^{\Delta t} \quad (9)$$

$$= \frac{T_i}{2} q_i (1 - e^{-2\Delta t/T_i}) \quad (10)$$

Now as seen in equation (4) above, for the steady state Markov process x_i , the magnitude of the PSD of the white driving noise is q_i , which is related to the steady state covariance p_i and the correlation time T_i according to:

$$q_i = 2p_i/T_i$$

Substituting this into equation (10) gives, for $i=1,2,3$:

$$Q_k(i,i) = p_i (1 - e^{-2\Delta t/T_i}) \quad (11)$$

We will now restrict our attention to the remaining 5-block. Since this is non-diagonal, it requires a non-trivial UDU^T decomposition. Henceforth for simplicity Q , Q_k and Φ shall refer to the corresponding 5-blocks rather than the full matrices. Thus, from (7) we have:

$$\Phi Q \Phi^T = \begin{pmatrix} e^{-\tau/T_4} & 0 & 0 & 0 & 0 \\ T_4(1-e^{-\tau/T_4}) & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-\tau/T_6} & 0 & 0 \\ 0 & -\tau S \sin \theta & \tau S \cos \theta & 1 & 0 \\ 0 & \tau S \cos \theta & \tau S \sin \theta & 0 & 1 \end{pmatrix} \begin{pmatrix} q_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Phi^T \quad (12)$$

$$= \begin{pmatrix} q_4 e^{-\tau/T_4} & 0 & 0 & 0 & 0 \\ q_4 T_4 (1 - e^{-\tau/T_4}) & 0 & 0 & 0 & 0 \\ 0 & 0 & q_6 e^{-\tau/T_6} & 0 & 0 \\ 0 & 0 & \tau q_6 S \cos \theta & 0 & 0 \\ 0 & 0 & \tau q_6 S \sin \theta & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-\tau/T_4} & T_4 (1 - e^{-\tau/T_4}) & 0 & 0 & 0 \\ 0 & 1 & 0 & -\tau S \sin \theta & \tau S \cos \theta \\ 0 & 0 & e^{-\tau/T_6} & \tau S \cos \theta & \tau S \sin \theta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (13)$$

=

$$\begin{pmatrix} q_4 e^{-2\tau/T_4} & q_4 T_4 (1 - e^{-\tau/T_4}) e^{-\tau/T_4} & 0 & 0 & 0 \\ q_4 T_4 (1 - e^{-\tau/T_4}) e^{-\tau/T_4} & q_4 T_4^2 (1 - e^{-\tau/T_4})^2 & 0 & 0 & 0 \\ 0 & 0 & q_6 e^{-2\tau/T_6} & \tau q_6 S \cos \theta e^{-\tau/T_6} & \tau q_6 S \sin \theta e^{-\tau/T_6} \\ 0 & 0 & \tau q_6 S \cos \theta e^{-\tau/T_6} & q_6 S^2 \cos^2 \theta \tau^2 & q_6 S^2 \sin \theta \cos \theta \tau^2 \\ 0 & 0 & \tau q_6 S \sin \theta e^{-\tau/T_6} & q_6 S^2 \sin \theta \cos \theta \tau^2 & q_6 S^2 \sin^2 \theta \tau^2 \end{pmatrix} \quad (14)$$

It is now clear that this Q_k also has a block diagonal form, with a 2-block and a 3-block. Assuming that the time dependent terms (speed S and heading θ) are constant over the integration interval, the above matrix can be explicitly integrated (as in equation (6)). For convenience we will label the individual terms as follows:

$$Q_k \equiv \begin{pmatrix} q_{11} & q_{12} & 0 & 0 & 0 \\ q_{21} & q_{22} & 0 & 0 & 0 \\ 0 & 0 & q_{33} & q_{34} & q_{35} \\ 0 & 0 & q_{43} & q_{44} & q_{45} \\ 0 & 0 & q_{53} & q_{54} & q_{55} \end{pmatrix} \quad (15)$$

where by symmetry $q_{ij} = q_{ji}$. Then q_{11} can be found as in equations (8) to (11) above:

$$q_{11} = \int_0^{\Delta t} q_4 e^{-2\tau/T_4} d\tau \quad (16)$$

$$\begin{aligned}
&= \frac{-T_4}{2} q_4 e^{-2\tau/T_4} \Big|_0^{\Delta t} \\
&= \frac{T_4}{2} q_4 (1 - e^{-2\Delta t/T_4})
\end{aligned} \tag{17}$$

$$= p_4 (1 - e^{-2\Delta t/T_4}) \tag{18}$$

Similarly for the other components:

$$\begin{aligned}
q_{12} &= \int_0^{\Delta t} T_4 q_4 (1 - e^{-\tau/T_4}) e^{-\tau/T_4} d\tau \\
&= \frac{T_4^2}{2} q_4 (1 - e^{-\tau/T_4})^2 \Big|_0^{\Delta t} \\
&= \frac{T_4^2}{2} q_4 (1 - e^{-\Delta t/T_4})^2 \\
&= T_4 p_4 (1 - e^{-\Delta t/T_4})^2
\end{aligned} \tag{19}$$

$$\begin{aligned}
q_{33} &= \int_0^{\Delta t} q_6 e^{-2\tau/T_6} d\tau \\
&= p_6 (1 - e^{-2\Delta t/T_6}) \quad (\text{as in } q_{11})
\end{aligned} \tag{20}$$

$$\begin{aligned}
q_{43} &= \int_0^{\Delta t} q_6 S \cos \theta \tau e^{-\tau/T_6} d\tau \\
&= q_6 S \cos \theta \frac{e^{-\tau/T_6}}{(1/T_6)^2} (-\tau/T_6 - 1) \Big|_0^{\Delta t} \\
&= -T_6 q_6 S \cos \theta [e^{-\Delta t/T_6} (\Delta t + T_6) - T_6] \\
&= 2T_6 p_6 S \cos \theta [1 - e^{-\Delta t/T_6} (1 + \Delta t/T_6)] \quad (21)
\end{aligned}$$

Similarly

$$\begin{aligned}
q_{53} &= \int_0^{\Delta t} q_6 S \sin \theta \tau e^{-\tau/T_6} d\tau \\
&= 2T_6 p_6 S \sin \theta [1 - e^{-\Delta t/T_6} (1 + \Delta t/T_6)] \quad (22)
\end{aligned}$$

$$\begin{aligned}
q_{54} &= \int_0^{\Delta t} q_6 S^2 \sin \theta \cos \theta \tau^2 d\tau \\
&= q_6 S^2 \sin \theta \cos \theta \frac{\tau^3}{3} \Big|_0^{\Delta t} \\
&= q_6 S^2 \sin \theta \cos \theta \frac{\Delta t^3}{3} \quad (23)
\end{aligned}$$

$$q_{44} = \int_0^{\Delta t} q_6 S^2 \cos^2 \theta \tau^2 d\tau$$

$$\begin{aligned}
&= q_6 s^2 \cos^2 \theta \frac{\tau^3}{3} \Big|_0^{\Delta t} \\
&= q_6 s^2 \cos^2 \theta \frac{\Delta t^3}{3}
\end{aligned} \tag{24}$$

Similarly

$$\begin{aligned}
q_{55} &= \int_0^{\Delta t} q_6 s^2 \sin^2 \theta \tau^2 d\tau \\
&= q_6 s^2 \sin^2 \theta \frac{\tau^3}{3} \Big|_0^{\Delta t} \\
&= q_6 s^2 \sin^2 \theta \frac{\Delta t^3}{3}
\end{aligned} \tag{25}$$

$$\begin{aligned}
q_{22} &= \int_0^{\Delta t} T_4^2 q_4 (1 - e^{-\tau/T_4})^2 d\tau \\
&= T_4^2 q_4 \int_0^{\Delta t} (1 - 2e^{-\tau/T_4} + e^{-2\tau/T_4}) d\tau \\
&= T_4^2 q_4 \left(\tau + 2T_4 e^{-\tau/T_4} - \frac{T_4}{2} e^{-2\tau/T_4} \right) \Big|_0^{\Delta t} \\
&= T_4^2 q_4 \left(\Delta t + 2T_4 e^{-\Delta t/T_4} - \frac{T_4}{2} e^{-2\Delta t/T_4} - 2T_4 + \frac{T_4}{2} \right) \\
&= T_4^3 q_4 \left(\Delta t/T_4 + 2e^{-\Delta t/T_4} - \frac{1}{2} e^{-2\Delta t/T_4} - \frac{3}{2} \right) \\
&= T_4^2 \left(\frac{2\Delta t}{T_4} + 4e^{-\Delta t/T_4} - e^{-2\Delta t/T_4} - 3 \right) p_4
\end{aligned} \tag{26}$$

Now by substituting equations (18) through (26) into equation (15) we can write the discrete 5×5 Q_k matrix as follows:

$$Q_k =$$

$$\begin{bmatrix} p_4(1-e^{-2\Delta t/T_4}) & T_4 p_4(1-e^{-\Delta t/T_4})^2 & 0 & 0 & 0 \\ T_4 p_4(1-e^{-\Delta t/T_4})^2 & T_4^2 p_4 \left(\frac{2\Delta t}{T_4} + 4e^{-\Delta t/T_4} - e^{-2\Delta t/T_4} \right) & 0 & 0 & 0 \\ 0 & 0 & p_6(1-e^{-2\Delta t/T_6}) & 2T_6 S \cos \theta A p_6 & 2T_6 S \sin \theta A p_6 \\ 0 & 0 & 2T_6 S \cos \theta A p_6 & \frac{2\Delta t^3}{3T_6} S^2 \cos^2 \theta p_6 & \frac{2\Delta t^3}{3T_6} S^2 \sin \theta \cos \theta p_6 \\ 0 & 0 & 2T_6 S \sin \theta A p_6 & \frac{2\Delta t^3}{3T_6} S^2 \sin \theta \cos \theta p_6 & \frac{2\Delta t^3}{3T_6} S^2 \sin^2 \theta p_6 \end{bmatrix} \quad (27)$$

where:

$$A = 1 - e^{-\Delta t/T_6} \left(1 - \frac{\Delta t}{T_6} \right) \quad (28)$$

As indicated in reference [1], the correlation times (T_4 and T_6) and steady state covariances (p_4 and p_6) for the Markov processes representing the error in the gyro drift rate and the odometer scale factor are assumed to be constants. Therefore it is easy to see how this process driving noise covariance matrix Q_k behaves numerically for different discretization intervals Δt . Using the values given in [2], we have for $\Delta t = 60$ seconds:

$$Q_k \cong \begin{bmatrix} 0.03p_4 & p_4 & 0 & 0 & 0 \\ p_4 & 40p_4 & 0 & 0 & 0 \\ 0 & 0 & 0.03p_6 & 237p_6 S \cos \theta & 237p_6 S \sin \theta \\ 0 & 0 & 237p_6 S \cos \theta & 40p_6 S^2 \cos^2 \theta & 40p_6 S^2 \sin \theta \cos \theta \\ 0 & 0 & 237p_6 S \sin \theta & 40p_6 S^2 \sin \theta \cos \theta & 40p_6 S^2 \sin^2 \theta \end{bmatrix} \quad (29)$$

and for $\Delta t = 1$ second:

$$Q_k \cong \begin{pmatrix} 0.0005p_4 & 0.0003p_4 & 0 & 0 & 0 \\ 0.0003p_4 & 0.001p_4 & 0 & 0 & 0 \\ 0 & 0 & 0.0005p_6 & 0.0003p_6S\cos\theta & 0.0003p_6S\sin\theta \\ 0 & 0 & 0.0003p_6S\cos\theta & 0.0002p_6S^2\cos^2\theta & 0.0002p_6S^2\sin\theta\cos\theta \\ 0 & 0 & 0.0003p_6S\sin\theta & 0.0002p_6S^2\sin\theta\cos\theta & 0.0002p_6S^2\sin^2\theta \end{pmatrix} \quad (30)$$

This now allows us to see the relative significance (or insignificance) of the off-diagonal terms.

4. UDU^T DECOMPOSITION OF Q_k

To decompose the Q_k matrix for use in one of the numerically superior "square root" formulations of the Kalman filter (see for example reference [3]), we follow Bierman in using the Cholesky factorization method. For this we must find matrices U and D such that U is upper triangular with 1's on the diagonal, D is diagonal and positive semi-definite, and

$$Q_k = UDU^T \quad (31)$$

Since Q_k is of block diagonal form, it can be easily shown that its square root has the same block diagonal form, and by simple extension so must its U factor (as is proven in Appendix A below). Therefore the factorization can be greatly simplified by performing it separately on the diagonal blocks of Q_k .

4.1. EXACT DECOMPOSITION OF THE 2X2 BLOCK

A general factorization for a 2x2 block can be found as follows. We equate the general matrix to the UDU^T product, where U and D are of the required form:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \quad (32)$$

$$= \begin{pmatrix} e & fd \\ 0 & f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (e+fd^2) & fd \\ fd & f \end{pmatrix} \quad (33)$$

We then solve for the unknown elements of U and D (e , f and d) as functions of the elements

of the general matrix (a, b and c). Therefore, by inspection, the exact solution is:

$$\begin{array}{lcl} f & = & c \\ d & = & b/c \\ e & = & a - b^2/c \end{array} \quad (34)$$

This can now be used to find the decomposition of the 2x2 block of Q_k , as given in (27). Thus we take a, b and c are from equation (27) and substitute into (34). The resulting expressions can be simplified as follows.

$$\begin{aligned} f &= c = Q_k(2,2) \\ &= p_4 T_4^2 \left[\frac{2\Delta t}{T_4} + 4e^{-\Delta t/T_4} - e^{-2\Delta t/T_4} - 3 \right] \end{aligned} \quad (35)$$

$$\begin{aligned} \therefore f &\cong p_4 T_4^2 \left\{ \frac{2\Delta t}{T_4} + 4 \left[1 - \frac{\Delta t}{T_4} + \frac{(-\Delta t/T_4)^2}{2} + \frac{(-\Delta t/T_4)^3}{3!} \right] \right. \\ &\quad \left. - \left[1 - \frac{2\Delta t}{T_4} + \frac{(-2\Delta t/T_4)^2}{2} + \frac{(-2\Delta t/T_4)^3}{3!} \right] - 3 \right\} \end{aligned}$$

(where we have used the first 4 terms of the Maclaurin series expansion for e^x)

$$\begin{aligned} &= p_4 T_4^2 \left\{ \frac{2\Delta t}{T_4} + 4 \cdot \frac{\Delta t}{T_4} + 2 \left(\frac{\Delta t}{T_4} \right)^2 - \frac{2}{3} \left(\frac{\Delta t}{T_4} \right)^3 - 1 + \frac{\Delta t}{T_4} - 2 \left(\frac{\Delta t}{T_4} \right)^2 + \frac{4}{3} \left(\frac{\Delta t}{T_4} \right)^3 - 3 \right\} \\ &= p_4 T_4^2 \frac{2}{3} \left(\frac{\Delta t}{T_4} \right)^3 \\ &= \frac{2p_4 \Delta t^3}{3T_4} \end{aligned} \quad (36)$$

This will be a good approximation provided that the discretization interval Δt is significantly

less than the correlation time T_4 (which it will be in PLANS).

Now the next term can also be simplified by similarly using the Maclaurin expansion and ignoring the higher order terms in $\Delta t/T_4$:

$$\begin{aligned} d &= b/c \\ &= T_4 p_4 (1 - e^{-\Delta t/T_4})^2 / c \end{aligned} \quad (37)$$

$$\begin{aligned} \therefore d &\cong \frac{T_4 p_4 (1 - 1 + \Delta t/T_4)^2}{\left[\frac{2p_4 \Delta t^3}{3T_4} \right]} \\ &= \frac{T_4 (\Delta t/T_4)^2}{\left[\frac{2\Delta t^3}{3T_4} \right]} \\ &= \frac{3}{2\Delta t} \end{aligned} \quad (38)$$

Finally the third term can also be simplified:

$$\begin{aligned} e &= a - b^2/c \\ &= p_4 (1 - e^{-2\Delta t/T_4}) - \frac{T_4^2 p_4^2 (1 - e^{-2\Delta t/T_4})^4}{p_4 T_4^2 \left[\frac{2\Delta t}{T_4} + 4e^{-\Delta t/T_4} - e^{-2\Delta t/T_4} - 3 \right]} \quad (39) \\ &\cong p_4 \left(\frac{2\Delta t}{T_4} \right) - \frac{T_4^2 p_4^2 \left(\frac{\Delta t}{T_4} \right)^4}{\left[\frac{2p_4 \Delta t^3}{3T_4} \right]} \\ &= \frac{2p_4 \Delta t}{T_4} - \frac{3T_4^3 p_4 \Delta t}{2T_4^4} \end{aligned}$$

$$\begin{aligned}
&= \frac{p4\Delta t}{T4} (2 - 3/2) \\
&= \frac{p4\Delta t}{2T4}
\end{aligned} \tag{40}$$

In this case the exact solution of (34) is given by (35), (37) and (39), with a good approximation given by (36), (38) and (40):

$$\begin{aligned}
f &\cong \frac{2p4\Delta t^3}{3T4} \\
d &\cong \frac{3}{2\Delta t} \\
e &\cong \frac{p4\Delta t}{2T4}
\end{aligned} \tag{41}$$

4.2. EXACT DECOMPOSITION OF THE 3X3 BLOCK

The decomposition of the 3x3 block can be found in a similar way:

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & g \end{pmatrix} = \begin{pmatrix} 1 & h & i \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ i & j & 1 \end{pmatrix} \tag{42}$$

$$\begin{aligned}
&= \begin{pmatrix} k & hL & im \\ 0 & L & jm \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ i & j & 1 \end{pmatrix} \\
&= \begin{pmatrix} (k+h^2L+i^2m) & (hL+ijm) & im \\ (hL+ijm) & (L+j^2m) & jm \\ im & jm & m \end{pmatrix}
\end{aligned} \tag{43}$$

Therefore we have:

$$m = g \quad (44)$$

$$\begin{aligned} jm &= e \\ \Rightarrow j &= e/m \\ &= e/g \end{aligned} \quad (45)$$

$$\begin{aligned} im &= c \\ \Rightarrow i &= c/m \\ &= c/g \end{aligned} \quad (46)$$

$$\begin{aligned} L + j^2 m &= d \\ \Rightarrow L &= d - e^2/m \\ &= d - e^2/g \end{aligned} \quad (47)$$

$$\begin{aligned} hL + ijm &= b \\ \Rightarrow h &= (b - ijm)/L \\ &= \frac{b - ce/m}{d - e^2/g} \\ &= \frac{b - ce/g}{d - e^2/g} \end{aligned} \quad (48)$$

$$\begin{aligned} k &= a - h^2 L - i^2 m \\ &= a - \frac{(b - ce/g)^2}{d - e^2/g} - c^2/g \end{aligned} \quad (49)$$

Now when the actual values for a, b, c, d, e and g are substituted from (27) and (42), we obtain the decomposition of the 3x3 block of the PLANS state vector driving noise covariance matrix, as follows:

$$m = \frac{2\Delta t^3}{3T6} s^2 \sin^2 \theta p_6 \quad (50)$$

$$j = \cot \theta \quad (51)$$

$$\begin{aligned} i &= \frac{2T6 s \sin \theta \left(1 - e^{-\Delta t/T6} \left(1 - \frac{\Delta t}{T6} \right) \right) p_6}{\frac{2\Delta t^3}{3T6} s^2 \sin \theta \cos \theta p_6} \\ &= \frac{3T6^2}{s\Delta t^3 \sin \theta} \left(1 - e^{-\Delta t/T6} \left(1 - \frac{\Delta t}{T6} \right) \right) \end{aligned} \quad (52)$$

$$\begin{aligned} L &= d - e \cdot \cot \theta \\ &= 0 \end{aligned} \quad (53)$$

$$h = 0 \quad (54)$$

$$\begin{aligned} k &= p_6 (1 - e^{-2\Delta t/T6}) - \frac{\left[2T6 s \sin \theta \left(1 - e^{-\Delta t/T6} \left(1 - \frac{\Delta t}{T6} \right) \right) p_6 \right]^2}{\frac{2\Delta t^3}{3T6} s^2 \sin^2 \theta p_6} \\ &= p_6 (1 - e^{-2\Delta t/T6}) - 6 \left(\frac{T6}{\Delta t} \right)^3 \left(1 - e^{-\Delta t/T6} \left(1 - \frac{\Delta t}{T6} \right) \right)^2 p_6 \end{aligned} \quad (55)$$

When these values are substituted into equation (42) we see that the exact decomposition of 3x3 block of Q_k has the form:

$$Q_k = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & \cot \theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i & \cot \theta & 1 \end{bmatrix} \quad (56)$$

where:

$$\begin{aligned}
m &= \frac{2\Delta t^3}{3T_6} S^2 \sin^2\theta p_6 \\
i &= \frac{3T_6^2}{S\Delta t^3 \sin\theta} \left(1 - e^{-\Delta t/T_6} \left(1 - \frac{\Delta t}{T_6} \right) \right) \\
k &= p_6 (1 - e^{-2\Delta t/T_6}) - 6 \left(\frac{T_6}{\Delta t} \right)^3 \left(1 - e^{-\Delta t/T_6} \left(1 - \frac{\Delta t}{T_6} \right) \right)^2 p_6
\end{aligned} \tag{57}$$

Now unfortunately this solution does not satisfy the requirement that the diagonal elements (k , l and m) be non-negative. In particular it can be seen that k can be negative by substituting the model values for p_6 , T_6 and Δt into equation (57). This requirement is necessary in order to use the Modified Weighted Gram-Schmidt algorithm (described in reference [3]), which is used by PLANS to propagate the covariance matrix.

However, it is quite common to use a much rougher approximation for the discrete Q_k matrix than is used here. In fact it is common to use $Q\Delta t$ in place of the integral of equation (6). This yields a Q_k which is diagonal, and hence has a trivial UDU^T decomposition. What has been done for PLANS however, is to find a decomposition which represents most elements of Q_k exactly (including the diagonal terms) and approximates the others, as described in the next chapter. Although this is not entirely exact, it is much better than the usual approximation.

5. A GOOD APPROXIMATION FOR THE 3X3 DECOMPOSITION

First note that Q_k is not positive definite. (The rank of the continuous 5x5 Q matrix, as shown in the bottom right corner of equation (3), is obviously only two.) Therefore the Cholesky decomposition of Q_k does not necessarily exist (see for example reference [4]), as we have indeed discovered. Of course Q is positive semi-definite (since it is a covariance matrix), which is the more basic requirement for the Kalman filter equations. In order to use the more numerically stable algorithms however, a decomposable approximation to Q_k must be found.

(After determining that the exact UDU^T decomposition had a negative diagonal element, another decomposition was attempted: the LDL^T , which uses lower triangular rather than upper triangular matrices. This also (perhaps predictably?) produced a negative diagonal element.)

The following approximation was found by inspection:

$$\begin{aligned}
 & \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b\cos\theta \\ 0 & 0 & b\sin\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b\cos\theta & b\sin\theta \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & bccos\theta \\ 0 & 0 & bcsin\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b\cos\theta & b\sin\theta \end{pmatrix} \\
 &= \begin{pmatrix} a^2c & abc\cos\theta & abc\sin\theta \\ abc\cos\theta & b^2c\cos^2\theta & b^2c\cos\theta\sin\theta \\ abc\sin\theta & b^2c\cos\theta\sin\theta & b^2c\sin^2\theta \end{pmatrix} \quad (58)
 \end{aligned}$$

By comparing this to equation (27), we see that this already has the correct θ dependence. In fact we would have an exact solution if we could find an a , b and c to satisfy the following:

$$a^2c = p_6(1-e^{-2\Delta t/T_6}) \quad (59)$$

$$b^2c = p_6 \frac{2\Delta t^3}{3T_6} s^2 \quad (60)$$

$$abc = 2p_6 s T_6 \left[1 - e^{-\Delta t/T_6} \left(1 - \frac{\Delta t}{T_6} \right) \right] \quad (61)$$

This does not generally (i.e. for arbitrary values of p_6 , T_6 and Δt) have an exact solution, as can be seen by comparing (59)x(60) and (61)², which should both be equal to $(abc)^2$. However, by solving (59) and (60) exactly and approximating (61) we have:

$$\begin{aligned} c &= p_6 \\ a &= \sqrt{1 - e^{2\Delta t/T_6}} \\ b &= s\Delta t \sqrt{\frac{2\Delta t}{3T_6}} \end{aligned} \quad (62)$$

This gives an exact solution on the diagonal and latitude/longitude cross terms (the (5,4) and (4,5) components of this 5x5 block of Q_k). The terms which are approximated (the (3,4) and (3,5) components) have the correct sign and the correct θ dependence. Further analysis indicates that the approximated terms are smaller than the true terms, provided only that the propagation interval is sufficiently short:

$$\Delta t < \sqrt{12} T_6 \cong 200 \text{ minutes} \quad (63)$$

which will certainly be the case in PLANS. This can be seen by substituting (62) into (58) and comparing to (27). Thus this approximation for Q_k is certainly better than the simpler $Q\Delta t$ approximation.

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APPENDIX A: BLOCK DIAGONAL FORM PRESERVATION UNDER DECOMPOSITION

In this section we shall prove that a block diagonal matrix can be decomposed (without loss of generality) by decomposing its diagonal blocks. This will be very useful for implementation of Kalman filters, since state models are often of block diagonal form, with a separate block for each independent sensor or subsystem. Deriving explicit decompositions for the corresponding driving noise covariance matrices Q_k can then be greatly simplified.

We will first prove the result for a matrix with two blocks on its diagonal. The extension to the general case is a straightforward application of mathematical induction.

Consider the Cholesky UDU^T decomposition of a positive definite square matrix M , which has two diagonal blocks:

$$M = \begin{pmatrix} M1 & 0 \\ 0 & M2 \end{pmatrix} \quad (A1)$$

$$= \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} A^T & 0 \\ B^T & C^T \end{pmatrix} \quad (A2)$$

Where A and C are upper triangular matrices with one's on the diagonal, and D and E are diagonal matrices (with non-zero elements on the diagonal, since M is positive definite). Multiplying (A2) out we obtain:

$$\begin{aligned} M &= \begin{pmatrix} AD & BE \\ 0 & CE \end{pmatrix} \begin{pmatrix} A^T & 0 \\ B^T & C^T \end{pmatrix} \\ &= \begin{pmatrix} (ADA^T + BEB^T) & BEC^T \\ CEB^T & CEC^T \end{pmatrix} \end{aligned} \quad (A3)$$

Now for (A2) to be a decomposition of (A1), the off-diagonal blocks of (A3) must be zero. Thus:

$$CEB^T = 0 \quad (A4)$$

$$BEC^T = 0 \quad (A5)$$

Since C is upper triangular with 1's on the diagonal, and E is a diagonal matrix, then if we define:

$$F \equiv CE \quad (A6)$$

we can easily see that F is upper triangular with the (non-zero) elements of E on its diagonal. Then (A4) becomes:

$$FB^T = 0 \quad (A7)$$

$$\begin{pmatrix} e1 & x & \dots & x & x \\ 0 & e2 & \dots & x & x \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e(n-1) & x \\ 0 & 0 & \dots & 0 & en \end{pmatrix} B^T = 0 \quad (A8)$$

Close examination of (A8) gives us the desired result: Starting with the last row of (A8), we can see that the bottom row of B^T must be zero (since $e_n \neq 0$). Given that the bottom row of B^T is zero, then examination of the second last row of (A8) shows that the second last row of B^T also must be zero (since $e_{n-1} \neq 0$). This can be continued up the rows to show that all rows of B^T are zero. From (A2) we can then see that the Cholesky decomposition of M is block diagonal, with the same block form as M.

Now this can easily be generalized to a matrix N with more than two blocks by separating one block at a time as follows. Let M1 in (A1) be the top block of N, so that M2 contains all the remaining blocks. The theorem as it stands proves that M2 can be decomposed separately from M1. Now simply apply the theorem again to M2 to see that its top diagonal block (the second block of N) can be decomposed separately from the rest (the third and remaining blocks of N). This can clearly be repeated until all the blocks have been separated.

APPENDIX B: POSITION ERROR COVARIANCE PROPAGATION IN PLANS

The time dependence of the state vector error covariance matrix P , in the absence of measurement updates, is described by the covariance propagation equation (see for example reference [5]):

$$P_{t+\Delta t} = \Phi(t+\Delta t) P_t \Phi^T(t+\Delta t) + Q_k \quad (B1)$$

Since the latitude and longitude error estimates are the last two elements of the state vector, the position error covariance is described by the last two diagonal elements of the covariance matrix P . Thus we will examine the propagation of these last two elements of P . From (B1) we can see that this involves only the bottom two rows of Φ and their transpose (the last two columns of Φ^T). From equation (5) we can see that the first four columns of the last two rows of Φ are zero, and can therefore be ignored.

Equation (27) can be used to obtain the relevant elements of Q_k . Here the p_6 refers to the steady state covariance of the sixth state (the odometer scale factor error), as explained by equation (11). This is a constant which comes from the error model, and has a value of about 0.001 (dimensionless).

If we assign the Markov process covariances, $P(5,5)$ and $P(6,6)$, to their steady state values, then we would have a gyro heading error $P(5,5)$ of about $(0.1 \text{ radian})^2$ and an odometer scale factor error $P(6,6)$ of about $(1\%)^2$. This is what would be expected in the absence of measurements, and its reasonableness has been verified by simulation. The relevant portion of (B1) can then be written as follows, using (5) for the form of Φ and (30) for the form of Q_k :

$$P_{t+\Delta t} \cong \begin{bmatrix} a & b & 1 & 0 \\ b & -a & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.01 & 0 & d & e \\ 0 & 0.0001 & f & g \\ d & f & h & i \\ e & g & i & j \end{bmatrix} \begin{bmatrix} a & b \\ b & -a \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} k \cos^2 \theta & k \sin \theta \cos \theta \\ k \sin \theta \cos \theta & k \sin^2 \theta \end{bmatrix} \quad (B2)$$

where the position error covariance before propagation is:

$$h = P(7,7)_t \quad (B3)$$

$$j = P(8,8)_t \quad (B4)$$

The elements a and b of the propagation matrix Φ are shown in (5) to be:

$$a = -\Delta t S \sin\theta \quad (B5)$$

$$b = \Delta t S \cos\theta \quad (B6)$$

Assuming a propagation interval of $\Delta t = 1$ second, the relevant elements of the driving noise covariance Q_k can be found from equation (30), which implies that :

$$k = 0.0002 p_6 S^2 \quad (B7)$$

Multiplying (B2) out we obtain:

$$P_{t+\Delta t} \cong \begin{bmatrix} (0.01a + d) & (0.0001b + f) & (ad + bf + h) & (ae + bg + i) \\ (0.01b + e) & (-0.0001a + g) & (bd - af + i) & (be - ag + j) \end{bmatrix} \begin{bmatrix} a & b \\ b & -a \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + Q_k \quad (B8)$$

$$= \begin{bmatrix} (0.01a^2 + ad + 0.0001b^2 + bf + ad + bf + h) & x \\ x & (0.01b^2 + be + 0.0001a^2 - ag + be - ag + j) \end{bmatrix} + Q_k$$

Therefore, the position error covariance terms are:

$$P(7,7)_{t+\Delta t} \cong (0.01a^2 + 0.0001b^2 + 2(ad + bf) + h) + k \cos^2\theta \quad (B9)$$

$$= P(7,7)_t + 0.01a^2 + 0.0001b^2 + k \cos^2\theta + 2(ad + bf)$$

$$\cong P(7,7)_t + S^2(0.01 \sin^2\theta + 0.0001 \cos^2\theta + 0.000002 \cos^2\theta) + 2S(-\sin\theta d + \cos\theta f) \quad (B10)$$

$$P(8,8)_{t+\Delta t} \cong (0.01b^2 + 0.0001a^2 + 2(be - ag) + j)) + k\sin^2\theta \quad (B11)$$

$$= P(8,8)_t + 0.01b^2 + 0.0001a^2 + k\sin^2\theta + 2(be - ag)$$

$$\cong P(8,8)_t + S^2(0.01\cos^2\theta + 0.0001\sin^2\theta + 0.0000002\sin^2\theta) + 2S(\underline{\cos\theta}e + \underline{\sin\theta}g) \quad (B12)$$

Thus the position covariance *can* decrease in the absence of measurements, if the underlined terms in equations (B10) and (B12) are large enough in the negative sense. This will happen for certain values of heading θ and speed S , provided the d , e , f and g terms are not too small. Simulations have shown that these terms can be large enough to cause $P(7,7)$ and $P(8,8)$ to decrease, particularly in the absence of position measurements. The physical interpretation is that while the vehicle is heading back towards its point of origin the heading and speed errors start to cancel the "outbound" errors. In the absence of position update measurements (from GPS or Transit) the outbound position errors will of course be caused entirely by the heading and speed errors, and will therefore be statistically correlated to them through the cross covariances d , e , f and g .

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The PLANS (Primary Land Arctic Navigation System), developed at DREO, optimally integrates a directional gyro/gyrocompass, an odometer, a 3-axis strapdown magnetometer, a GPS receiver, a Transit receiver, a baroaltimeter and a digital terrain elevation map, for the purpose of navigating a land vehicle in the Canadian Arctic under potentially adverse conditions. This report derives the exact form of the discrete driving noise covariance matrix Q_k which is needed to propagate the covariance matrix in the Kalman filter used by PLANS. It is shown that the exact Q_k does not have a Cholesky UDU^T decomposition. However, a good approximation is shown to have the necessary decomposition for use in the Biermann-Agee-Turner formulation of the Kalman filter. This approximate decomposition is then found. A general result on the preservation of block diagonal form under UDU^T decomposition is also proven.

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